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Extrapolation method for solving weakly singular nonlinear Volterra integral equations of the second kind ☆

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Abstract

Based on a new generalization of discrete Gronwall inequality in [L. Tao, H. Yong, A generalization of discrete Gronwall inequality and its application to weakly singular Volterra integral equality of the second kind, J. Math. Anal. Appl. 282 (2003) 56–62], Navot's quadrature rule for computing integrals with the end point singularity in [I. Navot, A further extension of Euler–Maclaurin summation formula, J. Math. Phys. 41 (1962) 155–184] and a transformation in [P. Baratella, A. Palamara Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, J. Comput. Appl. Math. 163 (2004) 401–418], a new quadrature method for solving nonlinear weakly singular Volterra integral equations of the second kind is presented. The convergence of the approximation solution and the asymptotic expansion of the error are proved, so by means of the extrapolation technique we not only obtain a higher accuracy order of the approximation but also get a posteriori estimate of the error.

Keywords: Nonlinear weakly singular Volterra equation; The asymptotic expansion; A posteriori estimate

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1. Introduction

Many physical and engineering problems lead to analysis of the nonlinear weakly singular Volterra integral equation of the second kind:

$$\bar{u}(s) = \bar{y}(s) + \int_a^s \bar{k}\big(s, t, \bar{u}(t)\big)\,dt, \quad a \leqslant t \leqslant s \leqslant b, \tag{1}$$

here

$$\bar{k}\big(s, t, \bar{u}(t)\big) = (s-t)^{\alpha}\big(\ln|s-t|\big)^{\beta}\hat{k}\big(s, t, \bar{u}(t)\big), \quad -1 < \alpha \leqslant 0,\ \beta = 0 \text{ or } 1.$$

We assume that $\hat{k}(s, t, \bar{u}(t))$ satisfies Lipshitz condition, i.e., for fixed s, t with $a \leqslant t \leqslant s \leqslant b$ there is a positive constant L independent of s and t, such that

$$\big|\hat{k}(s, t, \bar{u}) - \hat{k}(s, t, v)\big| \leqslant L|\bar{u} - v|, \quad \forall \bar{u}, v \in (-\infty, \infty). \tag{2}$$

By means of Banach's fixed point theorem, Kershaw proved existence, uniqueness of solutions of Eq. (1) in [10]. Moreover, using Lagrange linear interpolation formula, Kershaw presented a collocation method for solving Eq. (1) and proved existence, uniqueness and convergence of the approximative solutions, but the order of the accuracy of his method is very low. Because the integral equation (1) not only has weakly singular kernel but also, in general, the derivative $\bar{u}'(s)$ of the solution $\bar{u}(s)$ is unbounded at the initial point of the interval of the integration (see [1,2, 4–7]), therefore there is difficulty for the numerical treatment of (1). Brunner pointed out in [4] that under quasi-uniform meshes the order of convergence of any polynomial spline collocation approximation is only $1 + \alpha$. Although he proved that under a suitably graded mesh the polynomial spline collocation approximation with the degree m may theoretically obtain the order m of accuracy, the optimal order cannot be obtained because of serious round-off errors. In order to overcome the difficulty caused by the derivative of the solution with weak singularity at the initial point $t = a$, Palamara Orsi [16] proposed to apply product integration methods to solve (1), where a Nyström method is used on a small interval $[a, c]$ and a step by step method is used on $[c, b]$. Lubich [13] suggested that fractional linear multistep methods can be used to solve (1), he proved that if the linear multistep method $\omega = (\rho, \sigma)$ is stable and consistent of order p, the error of the approximation has estimate $O(t^{\beta-1}h^p)$, that is, the accuracy of the approximation may be lower in the neighborhood at the initial point $a = 0$ because of the weak singularity of $\bar{u}'(t)$. Hu discussed a fractional power spline collocation method called β-polynomial discrete collocation method in [9] and proved superconvergence properties.

However, using a simple transformation, Baratella and Palamara Orsi proved in [3] that a linear weakly singular Volterra integral equation of the second kind can be transformed into an equation which is still weakly singular, but whose solution is as smooth as we like. In this paper we apply the transformation to nonlinear weakly singular Volterra integral equation of the second kind (1). For that purpose, under a change of variable

$$\gamma(t) = (t-a)^q + a, \tag{3}$$

(1) becomes

$$u(s) = y(s) + \int_a^s \big(\gamma(s) - \gamma(t)\big)^{\alpha}\big(\ln\big|\gamma(s) - \gamma(t)\big|\big)^{\beta}\hat{k}\big(\gamma(s), \gamma(t), u(t)\big)\gamma'(t)\,dt,$$
$$a \leqslant t \leqslant s \leqslant \gamma^{-1}(b), \tag{4}$$

where q is a positive integer and $u(s) = \bar{u}(\gamma(s))$, $y(s) = \bar{y}(\gamma(s))$. Now Eq. (4) can be expressed by

$$u(s) = y(s) + \int_a^s k^*\big(s, t, u(t)\big)\, dt, \tag{5}$$

where

$$k^*\big(s, t, u(t)\big) = (s-t)^\alpha \big[\ln|s-t| + \delta(s,t)\big]^\beta k\big(s, t, u(t)\big), \quad -1 < \alpha \leqslant 0,\ \beta = 0, 1, \tag{6}$$

with

$$\delta(s,t) = \begin{cases} \ln\left|\frac{\gamma(s)-\gamma(t)}{s-t}\right|, & t \neq s, \\ \ln|\gamma'(s)|, & t = s, \end{cases} \tag{7}$$

and

$$k\big(s, t, u(t)\big) = \begin{cases} \left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha \hat{k}(\gamma(s), \gamma(t), u(t))\gamma'(t), & t \neq s, \\ (\gamma'(s))^\alpha \hat{k}(\gamma(s), \gamma(t), u(t))\gamma'(t), & t = s. \end{cases} \tag{8}$$

Obviously $\delta(s,t)$ and $k(s,t,u(t))$ are smooth if $\hat{k}(s,t,\bar{u})$ is smooth and $k(s,t,u(t))$ satisfies Lipshitz condition, i.e., for fixed s, t with $a \leqslant t \leqslant s \leqslant \gamma^{-1}(b)$, there is a positive constant L independent of s, t, such that

$$\big|k(s,t,u) - k(s,t,v)\big| \leqslant L|u-v|, \quad \text{for } \forall u, v \in (-\infty, \infty). \tag{9}$$

If Eq. (1) is linear, then taking $u(t) = \bar{u}(\gamma(t))\gamma'(t)$, (5) will be simplified to a linear equation whose solution does not involve any singularity in the first derivation (see [3]). But this method is ineffective for the nonlinear equation (5). However, for a sufficiently large q we can assert that $k(s,t,u(t))$ can be smooth at $t = a$. In fact, assume that there is a constant μ with $\mu > -1$, such that $u'(t) = O((t-a)^\mu)$ as $t \to a$. From (8) we have

$$\begin{aligned} \frac{d}{dt}k\big(s,t,u(t)\big) &= \frac{d}{dt}\big[\hat{k}\big(\gamma(s),\gamma(t),u(t)\big)\big]\gamma'(t)\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha \\ &\quad + \hat{k}\big(\gamma(s),\gamma(t),u(t)\big)\frac{d}{dt}\left[\gamma'(t)\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha\right] \\ &= \hat{k}_t\big(\gamma(s),\gamma(t),u(t)\big)\big(\gamma'(t)\big)^2\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha \\ &\quad + \hat{k}_u\big(\gamma(s),\gamma(t),u(t)\big)u'(t)\gamma'(t)\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha \\ &\quad + \hat{k}\big(\gamma(s),\gamma(t),u(t)\big)\gamma''(t)\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^\alpha \\ &\quad - \alpha\hat{k}\big(\gamma(s),\gamma(t),u(t)\big)\gamma'(t)\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)^{\alpha-1} \\ &\quad \times \frac{-\gamma'(t)(s-t) + \gamma(s) - \gamma(t)}{(s-t)^2} \\ &= \begin{cases} O\big((t-a)^{q-2}\big) & \text{for } t \to a \text{ and } s > a,\ \text{if } q \geqslant 2, \\ O\big((t-a)^\mu\big) & \text{for } s = t \text{ and } t \to a,\ \text{if } \mu \geqslant 0 \text{ and } q = 1. \end{cases} \end{aligned}$$

Taking a positive integer k and $q = k + 1$, we derive

$$\left.\frac{d^j k(s,t,u(t))}{dt^j}\right|_{t=a} = 0, \quad j = 1, \dots, k-1,$$

and

$$\left.\frac{d^j k(s,t,u(t))}{dt^j}\right|_{t=a} = O(1). \tag{10}$$

Thus (5) can be solved by the standard spline collocation methods and the standard product integration methods with a high order of accuracy.

However, the spline collocation methods and the product integration methods require to perform a lot of computations to determine the integration weight coefficients in advance, e.g., by using a piecewise polynomial collocation with degree $m-1$ we have to get $Nm(m+1)/2$ weight coefficients by computing $Nm(m+1)/2$ weakly singular integrals, where $1/N$ is the step length (see [5]). Therefore it is not convenient to use the spline collocation methods and the product integration methods in practical applications. On the other hand, direct quadrature methods (DQM), without doing any computations of integrals for getting the weight coefficients, have been effectively applied to solve Volterra integral equations of the second kind with some smooth kernels (see [1,8,11,14]), but so far there are only few papers dealing with the quadrature methods to solve nonlinear weakly singular Volterra integral equations of the second kind. In [12] based on Navot's quadrature rule [15] we presented a new quadrature method of solving (1) and proved that the error is of order $O(h^{2+\alpha})$.

In this paper we focus on DQM and their extrapolations for solving (1). It is well known that Richardson extrapolation as an accelerating convergence technique has been applied to many fields in computational mathematics (see [11]). However, to our knowledge, this paper may be the first attempt to apply Richardson extrapolation to accelerating convergence for the weakly singular nonlinear Volterra integral equations of the second kind, where the generalization of the discrete Gronwall inequality (see [12]) plays a role in the proof.

In Section 2 we present an algorithm based on a modified rectangle rule. In Section 3 the convergence and the error estimation of the algorithm are proved. In Section 4 asymptotic expansions, extrapolations and a posteriori asymptotic error estimation are obtained. In Section 5 we give some numerical examples.

2. Algorithm

We first describe the quadrature formula of the integration with endpoint singularity and Euler–Maclaurin expansion given by Navot in [15]. Let

$$I(G) = \int_a^b G(x)\,dx = \int_a^b (b-x)^\alpha \bigl(\ln|b-x|\bigr)^\beta g(x)\,dx, \tag{11}$$

where $-1 < \alpha \leqslant 0$, $\beta = 0, 1$, and $G(x) = (b-x)^\alpha (\ln|b-x|)^\beta g(x)$, $g(x)$ is smooth on $[a,b]$. Take the step length $h = (b-a)/N$, and mesh points $x_i = a + ih$, $i = 0, \dots, N$, where N is a sufficiently large integer.

Lemma 1. [15] *If $g(x) \in C^{2m}[a,b]$, then the error of the modified trapezoidal rule*

$$Q_N(G) = \frac{h}{2}G(x_0) + h\sum_{i=1}^{N-1} G(x_i) - \bigl[-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^{\beta}\bigr]g(b)h^{1+\alpha} \tag{12}$$

of integral (11) *has the following asymptotic expansion*:

$$\begin{aligned} E_N(G) &= Q_N(G) - I(G) \\ &= \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} G^{(2j-1)}(a)h^{2j} \\ &\quad + \sum_{j=1}^{2m-1} (-1)^j \bigl[-\beta\zeta'(-\alpha - j) + \zeta(-\alpha - j)(\ln h)^{\beta}\bigr] g^{(j)}(b) h^{j+\alpha+1}/j! \\ &\quad + O\bigl(h^{2m}\bigr), \end{aligned} \tag{13}$$

where B_{2j}, $j = 1, \ldots, m-1$, are Bernoulli numbers, and $\zeta(x)$ is the Riemann-zeta function.

By Lemma 1, it is easy to derive that if $g(x) \in C^2[a,b]$, then

$$E_N(G) = O\bigl(h^{2+\alpha}|\ln h|^{\beta}\bigr). \tag{14}$$

Now we construct the discrete equations from (5), (6). Setting $s = x_i$ in (5), we get

$$u(x_i) = y(x_i) + \int_{x_0}^{x_i} (x_i - t)^{\alpha}\bigl(\ln|x_i - t| + \delta(x_i, t)\bigr)^{\beta} k\bigl(x_i, t, u(t)\bigr)\, dt. \tag{15}$$

Using the quadrature formula (12) we obtain the following discrete equations:

$$\begin{cases} u_0 = y(x_0), \\ u_i = y(x_i) + \dfrac{h}{2}(x_i - x_0)^{\alpha}\bigl(\ln|x_i - x_0| + \delta(x_i, x_0)\bigr)^{\beta} k(x_i, x_0, u_0) \\ \qquad + h\displaystyle\sum_{j=1}^{i-1} (x_i - x_j)^{\alpha}\bigl(\ln|x_i - x_j| + \delta(x_i, x_j)\bigr)^{\beta} k(x_i, x_j, u_j) \\ \qquad - \bigl[-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^{\beta} - \beta\zeta'(-\alpha)\delta(x_i, x_i)\bigr] k(x_i, x_i, u_i) h^{1+\alpha}, \\ \quad i = 1, \ldots, N. \end{cases} \tag{16}$$

Equations (16), as a nonlinear diagonal system of equations in u_j, $j = 0, 1, \ldots, N$, can be computed by the following iterative algorithm:

Algorithm 1 *(Modified rectangle quadrature method).*

Step 1. Take $\varepsilon > 0$ sufficiently small and $\tilde{u}_0 = y(x_0)$, $i := 1$.
Step 2. Set $u_i^0 = \tilde{u}_{i-1}$ and $m := 0$, compute u_i^{m+1} ($i \leqslant N$) by the following simple iteration:

$$\begin{aligned}u_i^{m+1} &= y(x_i) + \frac{h}{2}(x_i - x_0)^\alpha \big(\ln|x_i - x_0| + \delta(x_i, x_0)\big)^\beta k(x_i, x_0, \tilde{u}_0) \\ &\quad + h\sum_{j=1}^{i-1}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta k(x_i, x_j, \tilde{u}_j) \\ &\quad + \big[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta + \beta\zeta'(-\alpha)\delta(x_i, x_i)\big]h^{1+\alpha} k\big(x_i, x_i, u_i^m\big).\end{aligned} \tag{17}$$

Step 3. If $|u_i^{m+1} - u_i^m| \leqslant \varepsilon$, then set $\tilde{u}_i := u_i^{m+1}$ and $i := i + 1$, go to step 2; otherwise set $m := m + 1$, go to step 2.

3. Convergence and error estimate

Using integral formula (12), Eq. (15) can be written as follows:

$$\begin{aligned}u(x_i) &= y(x_i) + h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta k\big(x_i, x_j, u(x_j)\big) \\ &\quad + h w_{ii} k\big(x_i, x_i, u(x_i)\big) \\ &\quad + E_{i,t}\big((x_i - t)^\alpha \big(\ln|x_i - t| + \delta(x_i, t)\big)^\beta k\big(x_i, t, u(t)\big)\big),\end{aligned} \tag{18}$$

where

$$w_{i0} = \frac{1}{2}, \qquad w_{ii} = h^\alpha\big[\beta\zeta'(-\alpha) - \beta\zeta(-\alpha)\delta(x_i, x_i) + \zeta(-\alpha)(\ln h)^\beta\big],$$

and

$$w_{ij} = 1, \quad \text{for } 1 \leqslant j < i.$$

By Lemma 1, the remainder has estimate

$$\big|E_{i,t}\big((x_i - t)^\alpha \big(\ln|x_i - t| + \delta(x_i, t)\big)^\beta k\big(x_i, t, u(t)\big)\big)\big| = O\big(h^{2+\alpha}|\ln h|^\beta\big). \tag{19}$$

Setting $e_i = u(x_i) - u_i$ and subtracting (16) from (18), we obtain that errors $\{e_i\}$ satisfy equations

$$\begin{cases} e_0 = 0, \\ e_i = h\displaystyle\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta \big[k\big(x_i, x_j, u(x_j)\big) \\ \qquad - k(x_i, x_j, u_j)\big] + h w_{ii}\big[k\big(x_i, x_i, u(x_i)\big) - k(x_i, x_i, u_i)\big] \\ \qquad + E_{i,t}\big((x_i - t)^\alpha \big(\ln|x_i - t| + \delta(x_i, t)\big)^\beta k\big(x_i, t, u(t)\big)\big), \quad i = 1, \ldots, N. \end{cases} \tag{20}$$

However, for Volterra equation of the second kind with the weakly singular kernel, we cannot directly apply the discrete Gronwall inequality to prove the convergence as in the case with the continuous kernel. In [12] we proved a new generalization of discrete Gronwall inequality, which will be applied to prove the convergence and error estimate of the approximation. For convenience, here it is expressed as follows.

Lemma 2. [12] *Suppose sequence* $\{e_i\}_{i=1}^N$ *satisfies*

$$e_0 = 0, \qquad |e_i| \leqslant \sum_{j=1}^{i-1} B_{ij}|e_j| + A, \quad 1 \leqslant i \leqslant N,$$

where $B_{ij} = 2Lh(x_i, -x_j)^\alpha(\ln|x_i - x_j|)^\beta$, $-1 < \alpha \leqslant 0$, $\beta = 0, 1$, *and* h *is sufficiently small to have* $Lhw_{ii} \leqslant 1/2$. *Then*

$$|e_i| \leqslant HA, \quad \text{where } H = \sum_{k=0}^{\infty} \frac{R^k}{(k!)^s}, \; R = 2L(b-a)^s\Gamma(s)e^{\frac{1}{(12s)}}\left(\frac{e}{s}\right)^s, \; s = 1 + \alpha. \tag{21}$$

Theorem 1. *Assume that* h *is sufficiently small, then the solutions of the nonlinear discrete equations* (16) *exist and are unique, and the simple iteration* (17) *is geometrically convergent.*

Proof. First, if $\{u_i\}$ and $\{v_i\}$ are solutions of (16), then the differences $\{z_i = u_i - v_i\}$ satisfy the inequality

$$|z_i| \leqslant \sum_{j=1}^{i-1} \bar{B}_{ij}|z_j|, \quad 1 \leqslant i \leqslant N,$$

where $\bar{B}_{ij} = 2Lh(x_i - x_j)^\alpha(\ln|x_i - x_j| + |\delta(x_i, x_j)|)^\beta$. Since $\delta(s,t)$ is continuous, we can choose a constant $c > 0$ such that $|\delta(s,t)| \leqslant c|\ln|s - t||$ for $a \leqslant s \leqslant t \leqslant \gamma^{-1}(b)$. Letting $\bar{L} = (1 + c)L$, we derive

$$\bar{B}_{ij} \leqslant 2\bar{L}h(x_i - x_j)^\alpha\bigl(\ln|x_i - x_j|\bigr)^\beta. \tag{22}$$

Using inequality (21) with $A = 0$, we get that $z_j = 0$, $j = 0, \ldots, N$.

Secondly, from the simple iteration (17) we easily derive

$$\begin{aligned}\bigl|u_i^{n+1} - u_i^n\bigr| &\leqslant \bigl[\beta\zeta'(-\alpha) + \beta\zeta'(-\alpha)\delta(x_i, x_i) - \zeta(-\alpha)(\ln h)^\beta\bigr] \\ &\quad \times h^{1+\alpha}\bigl|k\bigl(x_i, x_i, u_i^n\bigr) - k\bigl(x_i, x_i, u_i^{n-1}\bigr)\bigr| \\ &\leqslant Lhw_{ii}\bigl|u_i^n - u_i^{n-1}\bigr| \leqslant \frac{1}{2}\bigl|u_i^n - u_i^{n-1}\bigr|,\end{aligned}$$

where we assume that h is so small that $Lhw_{ii} \leqslant 1/2$. Therefore we prove that the simple iteration (17) is geometrically convergent. Obviously, its limit is the unique solution of (16). □

In order to assert that Algorithm 1 is stable, we need to estimate the error $\{u_i - \tilde{u}_i\}$ as follows.

Theorem 2. *There is a positive constant* C *independent of* h *such that*

$$|u_i - \tilde{u}_i| \leqslant C\varepsilon h^{1+\alpha}|\ln h|^\beta, \tag{23}$$

where $\tilde{u}_i = u_i^{n+1}$ *is defined by Algorithm* 1.

Proof. Letting $v_i = u_i - \tilde{u}_i$ and subtracting (16) from (17), we get

$$\begin{cases} v_0 = 0, \\ v_i = h \sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta \big[k(x_i, x_j, u_j) - k(x_i, x_j, \tilde{u}_j)\big] \\ \qquad + h w_{ii}\big[k(x_i, x_i, u_i) - k\big(x_i, x_i, u_i^m\big)\big] \\ \quad = h \sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta \big[k(x_i, x_j, u_j) - k(x_i, x_j, \tilde{u}_j)\big] \\ \qquad + h w_{ii}\big[k(x_i, x_i, u_i) - k\big(x_i, x_i, u_i^{m+1}\big)\big] + h w_{ii}\big[k\big(x_i, x_i, u_i^{m+1}\big) - k\big(x_i, x_i, u_i^m\big)\big], \\ \quad i = 1, \ldots, N. \end{cases}$$

Thus from Lipshitz condition (9) and Algorithm 1 we derive the inequality

$$|v_i| \leqslant \sum_{j=0}^{i-1} \bar{B}_{ji}|v_j| + 2Lhw_{ii}\varepsilon, \quad i = 1, \ldots, N.$$

Using Lemma 2, we get estimate (23). □

Theorem 3. *There is a positive constant c independent of h such that errors $e_j = u(x_j) - u_j$, $j = 0, \ldots, N$, have the estimate*

$$\max_{0 \leqslant i \leqslant N} |e_i| \leqslant c h^{2+\alpha} |\ln h|^\beta. \tag{24}$$

Proof. From (20) we easily derive that $\{e_i\}$ satisfy the inequality

$$\begin{cases} e_0 = 0, \\ |e_i| \leqslant A + \sum_{j=1}^{i-1} \bar{B}_{ij}|e_j|, \end{cases}$$

where $A = \max_{1 \leqslant i \leqslant N} \max_{a \leqslant t \leqslant b} |2E_{i,t}((x_i - t)^\alpha (\ln|x_i - t| + \delta(x_i, t))^\beta k(x_i, t, u(t)))|$ and $\bar{B}_{ij} = 2Lh(x_i - x_j)^\alpha (\ln|x_i - x_j| + \delta(x_i, x_j))^\beta$. By using estimates (19), (22) and Lemma 2, Theorem 3 is proved. □

4. Asymptotic expansion and extrapolation

In this section we shall prove that the errors have asymptotic expansions. By Lemma 1, equality (15) has a high order asymptotic expansion

$$\begin{aligned} u(x_i) = y(x_i) &+ h \sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i, x_j)\big)^\beta k\big(x_i, x_j, u(x_j)\big) \\ &+ h w_{ii} k\big(x_i, x_i, u(x_i)\big) + T_0(x_i) h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i, x_i)\big)^\beta \\ &+ T_1(x_i) h^{2+\alpha} + T_2(x_i) h^2 + O\big(h^{3+\alpha}\big), \quad i = 1, \ldots, N, \end{aligned} \tag{25}$$

where

$$T_0(s) = -\zeta(-\alpha - 1)\frac{d}{dt}k\big(s, t, u(t)\big)\big|_{t=s},$$

$$T_1(s) = \beta\zeta'(-\alpha - 1)\frac{d}{dt}k\big(s, t, u(t)\big)\big|_{t=s},$$

and

$$T_2(s) = -\frac{B_2}{2}\frac{d}{dt}k\big(s,t,u(t)\big)\Big|_{t=a}.$$

Subtracting (16) from (25), we get

$$\begin{aligned}
e_i &= u(x_i) - u_i \\
&= h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i,x_j)\big)^\beta \big[k\big(x_i,x_j,u(x_j)\big) - k(x_i,x_j,u_j)\big] \\
&\quad + hw_{ii}\big[k\big(x_i,x_i,u(x_i)\big) - k(x_i,x_i,u_i)\big] + T_0(x_i)h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^\beta \\
&\quad + T_1(x_i)h^{2+\alpha} + T_2(x_i)h^2 + O\big(h^{3+\alpha}\big) \\
&= h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j| + \delta(x_i,x_j)\big)^\beta k_u\big(x_i,x_j,u(x_j)\big)e_j \\
&\quad + hw_{ii}k_u\big(x_i,x_i,u(x_i)\big)e_i + T_0(x_i)h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^\beta \\
&\quad + T_1(x_i)h^{2+\alpha} + T_2(x_i)h^2 + O\big(h^{3+\alpha}\big).
\end{aligned} \tag{26}$$

Obviously, $T_1(s) \equiv 0$ if $\beta = 0$, and $T_2(s) = 0$ if $q > 2$.

For given $u(t)$ we construct three auxiliary linear Volterra integral equations of the second kind:

$$T_k^*(s) = T_k(s) + \int_a^s k_u\big(s,t,u(t)\big)(s-t)^\alpha\big[\ln|s-t| + \delta(s,t)\big]^\beta T_k^*(t)\,dt, \quad k = 0,1,2,$$

and their approximation equations:

$$\begin{aligned}
T_{k,i}^{*h} &= T_k(s_i) + h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha\big(\ln|x_i - x_j| + \delta(x_i,x_j)\big)^\beta k_u\big(s_i,x_j,u(x_j)\big)T_{k,j}^{*h} \\
&\quad + hw_{ii}k_u\big(s_i,x_i,u(x_i)\big)T_{k,i}^{*h}, \quad i = 0,\dots,N,\ k = 0,1,2,
\end{aligned} \tag{27}$$

where $k_u(s,t,u) = \frac{\partial}{\partial u}k(s,t,u)$. Using Theorem 3, we get

$$\max_{0\leqslant i\leqslant N}\big|T_k^*(s_i) - T_{k,i}^{*h}\big| = O\big(h^{2+\alpha}\big), \quad k = 0,1,2.$$

Putting (27) into (26), we have

$$\begin{aligned}
& e_i - h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^\beta T_{0,i}^{*h} - h^{2+\alpha}T_{1,i}^{*h} - h^2T_{2,i}^{*h} \\
&\quad = h\sum_{j=0}^{i-1} w_{ij}(s_i - x_j)^\alpha k_u\big(s_i,x_j,u(x_j)\big)\big(e_j - h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^\beta T_{0,i}^{*h} \\
&\qquad - h^{2+\alpha}T_{1,j}^{*h} - h^2T_{2,i}^{*h}\big) + hw_{ii}k_u\big(s_i,x_i,u(x_i)\big) \\
&\qquad \times \big(e_i - h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^\beta T_{0,i}^{*h} - h^{2+\alpha}T_{1,i}^{*h} - h^2T_{2,i}^{*h}\big) + O\big(h^{3+\alpha}\big), \\
&\quad i = 1,\dots,N.
\end{aligned} \tag{28}$$

Let

$$E_i = e_i - h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^{\beta} T_{0,i}^{*h} - h^{2+\alpha} T_{1,i}^{*h} - h^2 T_{2,i}^{*h};$$

then (28) becomes

$$E_i = h\sum_{j=0}^{i-1} w_{ij}(s_i - t_j)^{\alpha} k_u\big(s_i, x_j, u(x_j)\big)E_j + h w_{ii} k_u\big(s_i, x_i, u(x_i)\big)E_i + O\big(h^{3+\alpha}\big),$$
$$i = 1,\dots,N.$$

Using Lemma 2, we derive

$$\begin{aligned} E_i &= e_i - h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^{\beta} T_{0,i}^{*h} - h^{2+\alpha} T_{1,i}^{*h} - h^2 T_{2,i}^{*h} \\ &= O\big(h^{3+\alpha}\big), \quad i = 1,\dots,N. \end{aligned}$$

Replacing $T_{k,i}^{*h}$ by $T_k^*(s_i)$ $(k = 0,1,2)$, respectively, we get

$$\begin{aligned} u_i = u(s_i) &- h^{2+\alpha}\big(\ln h - \beta\zeta'(-\alpha)\delta(x_i,x_i)\big)^{\beta} T_0^*(s_i) - h^{2+\alpha} T_1^*(s_i) - h^2 T_2^*(s_i) \\ &+ O\big(h^{3+\alpha}\big), \quad i = 1,\dots,N; \end{aligned} \tag{29}$$

that is, from the above results we have proved the following theorem.

Theorem 4. *If solution u of* (5) *belongs to $C^2(a, \gamma^{-1}(b))$, and for fixed s and t, $k(s,t,u(t))$ has second order partial derivatives with respect to u as well as both $k(s,t,u(t))$ and $k_u(s,t,u(t)) = \frac{\partial}{\partial u}k(s,t,u(t))$ satisfy Lipshitz condition* (9)*, then there are functions $T_k^*(s)$, $k = 0,1,2$, independent of h, such that the asymptotic expansion* (29) *is obtained.*

Obviously, $T_1^*(s) \equiv 0$ if $\beta = 0$. It implies that for no logarithmic singular case $(\beta = 0)$ Richardson $h^{2+\alpha}$-extrapolation

$$u_1^*(s_i) = \frac{2^{2+\alpha} u_i^{h/2} - u_i^h}{2^{2+\alpha} - 1} = u(s_i) + O\big(h^2\big), \quad i = 1,\dots,N, \tag{30}$$

has accuracy $O(h^2)$. In order to obtain accuracy $O(h^{3+\alpha})$, we should continue to use Richardson h^2-extrapolation. For logarithmic singular case, if $\beta = 1$ and $\alpha = 0$, Richardson h^2-extrapolation only obtains accuracy $O(h^2)$, we should continue to use Richardson h^2-extrapolation to obtain accuracy $O(h^{3+\alpha})$ again. If $\beta = 1$ and $\alpha > 0$, then Richardson $h^{2+\alpha}$-extrapolation only has accuracy $O(h^{2+\alpha})$ (see [11]). To obtain accuracy $O(h^2)$, we should continue to use Richardson $h^{2+\alpha}$-extrapolation again. The second time Richardson $h^{2+\alpha}$-extrapolation can be implemented as follows. First compute (30), then replacing h by $h/2$ gives

$$u_2^*(s_i) = \frac{2^{2+\alpha} u_i^{h/4} - u_i^{h/2}}{2^{2+\alpha} - 1}, \quad i = 1,\dots,N. \tag{31}$$

At last

$$u^*(s_i) = \frac{2^{2+\alpha} u_2^*(s_i) - u_1^*(s_i)}{2^{2+\alpha} - 1}, \quad i = 1,\dots,N. \tag{32}$$

Theorem 4 ensures that $u^*(s_i)$ has accuracy $O(h^2)$. To obtain accuracy $O(h^{3+\alpha})$ we may implement Richardson h^2-extrapolation again. Moreover, for both the case $\beta = 0$ or the case $\beta = 1$ but $\alpha = 0$, a posteriori asymptotic error estimate

$$\begin{aligned}
\left|u_i^{h/2} - u(s_i)\right| &= \left|\frac{2^{2+\alpha} u_i^{h/2}}{2^{2+\alpha} - 1} - \frac{u_i^h}{2^{2+\alpha} - 1} - u(s_i)\right| \\
&= \left|\frac{2^{2+\alpha} u_i^{h/2} - u_i^h}{2^{2+\alpha} - 1} - u(s_i) + \frac{u_i^h - u_i^{h/2}}{2^{2+\alpha} - 1}\right| \\
&\leqslant \left|u_1^*(s_i) - u(s_i)\right| + \left|\frac{u_i^h - u_i^{h/2}}{2^{2+\alpha} - 1}\right| = \left|\frac{u_i^h - u_i^{h/2}}{2^{2+\alpha} - 1}\right| + O\left(h^2\right)
\end{aligned}$$

can be derived by Theorem 4, i.e., we can bound the error $|u_i^{h/2} - u(s_i)|$ by $|(u_i^h - u_i^{h/2})/(2^{2+\alpha} - 1)|$, which is important to construct adaptable algorithms.

5. Numerical examples

Example 1. Consider the linear Volterra integral equation with algebraic singularity presented in [9]:

$$\varphi(s) = \frac{1}{2}\pi s + \sqrt{s} + \int_0^s -\frac{\varphi(t)}{\sqrt{s-t}}\, dt, \quad 0 \leqslant t \leqslant s \leqslant 1, \tag{33}$$

with the exact solution $\varphi(s) = \sqrt{s}$. For transformation $\gamma(t) = t^2$, the errors of the approximations and their Richardson extrapolations at $s = 1.0$ are shown in Table 1.

Note that although the accuracy obtained in [9] is better than that of our algorithm, the computational complexity is larger than ours. Moreover, it is important that we can use extrapolation here to get a high accuracy.

Example 2. Consider the Volterra integral equation with logarithmic singularity:

$$\varphi(s) = g(s) - \int_{-1}^s \ln|s-t|\varphi(t)\, dt, \quad -1 \leqslant t \leqslant s \leqslant 1, \tag{34}$$

with

$$g(s) = \sqrt{s+1} + \frac{4}{3}\sqrt{s+1}(s+1)\ln\left(2\sqrt{s+1}\right) - \frac{16}{9}\sqrt{s+1}(s+1),$$

where the exact solution is $\varphi(s) = \sqrt{s+1}$. For $\gamma(t) = \frac{1}{2}(t+1)^2 - 1$, the errors of the approximations and their Richardson extrapolations at $s = 1.0$ are shown in Table 2.

Compare with results of Tables 4 and 8 in [3], where for Nyström method with $N = 8$ and $q = 2$ the relative error is 6.7E–8 and for Simpson's product integration with $N = 200$ and $q = 2$ is 4.4E–10. Obviously both results are much better than the result of Table 2, but their

Table 1
The errors for algebraic singularity equation at $s = 1.0$

N	Error	A posteriori estimate	$h^{3/2}$-extrapolation	h^2-extrapolation	In [9]
10	5.281E–3				1.97E–5
20	1.87E–3	1.865E–3	4.446E–6		3.74E–6
40	6.62E–4	6.606E–4	1.322E–6	2.81E–7	7.10E–7

Table 2
The errors for logarithmic singularity equation at $s = 1.0$

N	Error	A posteriori estimate	h^2-extrapolation	h^2-extrapolation again
50	9.4596E–3			
100	2.7185E–3	2.247E–3	4.7260E–4	
200	7.6955E–4	6.495E–4	1.2043E–4	3.48E–6

Table 3
The errors for nonlinear equation at $s = 1.0$

h	Error	A posteriori estimate	h^2-extrapolation	h^2-extrapolation again
0.1	1.263E–3			
0.05	3.65E–4	2.993E–4	6.56E–5	
0.025	1.03E–4	8.73E–5	1.56E–5	1.06E–6

Table 4
The errors for algebraic singularity equation at $t = 8$ and $q = 1$

h	Error	A posteriori estimate	$h^{3/2}$-extrapolation	h^2-extrapolation	In [13]
0.1	7.3523E–4				1.07E–5
0.05	2.5990E–4	2.5996E–4	6.5833E–8		8.19E–7
0.025	9.1897E–5	9.1883E–5	1.4179E–8	3.0388E–9	3.02E–8

computational complexity is larger than our algorithm, e.g., for Nyström method a lot of work must be done to compute the zeros $\{x_i,\ i = 1, \dots, N\}$ of the Jacobi polynomial $P_N^{(\alpha,0)}(x)$ and the weighting coefficients

$$w_{n,j} = \int_{-1}^{x_n} (x_n - s)^{-\alpha} I_{N,j}(s)\, ds, \quad n = 0, \dots, N,\ j = 0, \dots, N.$$

Moreover our method is in common use.

Example 3. Consider the nonlinear Volterra integral equation with logarithmic singularity

$$\varphi(s) = -\frac{1}{2}s^2 \ln s + \frac{3}{4}s^2 + \sqrt{s} + \int_0^s \ln|s - t|\varphi^2(t)\, dt, \quad -1 \leqslant t \leqslant s \leqslant 1, \tag{35}$$

with the exact solution $\varphi(s) = \sqrt{s}$. For $q = 1$, the errors of the approximations and their Richardson extrapolations at $s = 1.0$ are shown in Table 3.

Example 4. Consider the nonlinear Abel–Volterra integral equation with algebraic singularity in [13,16]

$$y(t) = -\frac{1}{\sqrt{\pi}} \int_0^t (t - s)^{-1/2} \big(y(s) - \sin(s)\big)^3\, ds, \tag{36}$$

with the exact solution $y(8) = 0.3236412904$ and $y(t) = O(t^{3.5})$ as $t \to 0$ (see [16]). The errors of the approximations and their Richardson extrapolations at $t = 8.0$ are shown in Table 4.

The tables show that our quadrature methods not only possess high accuracy and low computational complexity, but also extrapolations and a posterior estimation are very effective.

Remark. Comparing DQM and other methods [1,3–6,9,13,16], we may conclude that the computational complexity of DQM is lower than others, because Algorithm 1 is just a simple iteration without doing any integral calculations. On the other hand, by other methods a lot of work must be done to compute coefficients of the discrete matrix by means of integral calculations, whose work could be large enough to exceed that of solving the discrete equation. Moreover, by the extrapolation method we not only obtain a high order of accuracy, but also a posteriori error estimate is conveniently derived, which means that we are sure to get a satisfactory approximation by an adaptive process.

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